
The Kinematics of Machines

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II. *The Kinematics of Machines.*

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WHATEVER machine be examined, it will be found to consist of a number of pieces, each of which is connected with one or more other pieces, in such a way as to be capable of some kind of motion relatively to those pieces.

The nature of the relative motion will depend largely on the form of the surfaces of mutual contact, but also on other influences. In some cases it will be of a simple character, in others, more complex.

It is proposed to refer to these as the *elementary mechanical motions*, and to regard a machine as an embodiment of a number of such motions which, together, provide that particular kind of, more or less complicated, movement which is required to serve the purpose of the machine.

The principal object of this paper is to indicate certain geometrical laws which govern the association of these elementary mechanical motions in the composition of machines, and it will be shown that in the examination of the influence of these laws one is led to a systematic classification of all machines, and to the enumeration of an exhaustive list of some, if not all, classes of simple machines.

Further, the idea that a machine is an embodiment of a combination of *elementary motions* (of which it will be found that the number of kinds is comparatively limited), enables one, by adopting a suggestive symbol for each, to indicate the composition of every machine movement by means of a simple formula.

If the machine consists of only two separately moving pieces then only one elementary motion will be involved in the construction.

If there are three pieces, V, X, and Z, for example, V and X may be joined to and in contact with Z, but not connected with one another. In this case the motion of V relatively to Z will be quite independent of the motion of X, and the apparatus should be regarded as a compound mechanism having one link, Z, common to two simple mechanisms.

If V and X are connected together, then, under some circumstances, relative motion of the three pieces will be possible, and there will be three motions included in the mechanism, which will be related to one another.

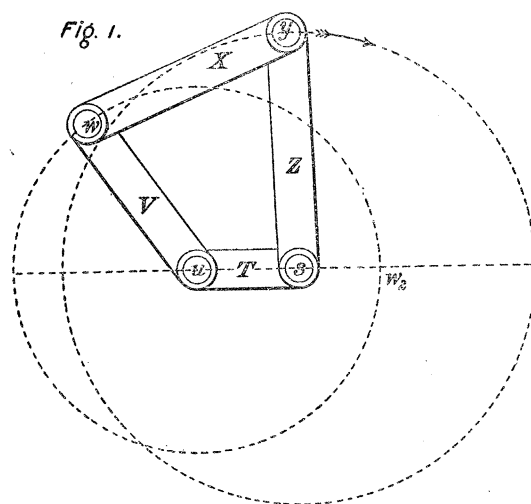
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If, in the case in which the pieces move in or parallel to one plane, the links are connected together in one continuous chain, the last being joined to the first, the number of links and motions will often be required to be not less than four to permit of any movement, and, if the number so joined exceeds four, the movement will be indeterminate and the linkage will not be suitable to serve the purpose of a machine in which a definite movement is required.

The general case of a simple plane mechanism is represented in fig. 1, in which four links are united together in one continuous chain by four pins, the axes of which are parallel to one another. If three links only were joined together in this way, they would form a rigid triangle, whereas, if five were so joined the relative movement would be indeterminate.

It will be seen further on that, by imagining the lengths of the links to be suitably changed up to the limits of zero and infinity, this mechanism may be made to move in all the ways which are possible for a simple machine, the parts of which move in or parallel to one plane, also by supposing other changes to occur the same mechanism will be seen to be representative of other classes of mechanisms in which the parts do not move in or parallel to one plane. P. 31.

It will be shown later, p. 36, that machines which consist of more than four links are compound mechanisms, which may be analysed into two or more simple ones, in which one or more links are common to two or more of the simple mechanisms which together compose the compound machine.



Simple Plane Mechanisms.

If, in the mechanism of fig. 1, the links are all of finite length, each link will be capable of moving relatively to an adjacent one, either by turning completely and continuously around, which will be referred to as a turning motion and represented by the letter O, or by swinging through a part of a revolution and reciprocating to and

fro. Such a motion will be referred to as a swinging motion, and represented by U. Which of these two elementary motions occur at any joint will depend on the proportions which the lengths of the links bear to one another.

There are two geometrical laws which govern the association of these two motions in this mechanism, and which also apply to other derived mechanisms.

Law I. *The sum of the four angles of the plane quadrilateral, $s u w y$ of fig. 1, is constant.*

Law II. has to do with the proportions necessary to admit of the complete rotation of one link relatively to the next in sequence. It rests on the established fact that one side of a triangle cannot be greater than the sum of the other two. The expression of this law will be given later.

In fig. 1 suppose the links V and Z to be capable of turning continuously around relatively to T in the direction of the arrow, then, with every revolution, the angle at s may be regarded as having been increased by the amount of 2π , and when, in the movement, the angle between the two links V and T is reduced to zero by w coinciding with w_2 , further movement in the same direction may be regarded as causing the angle to become negative.

Regarded in this way Law I. will still hold good, the sum of the four angles estimated from any initial position will be constant.

It will follow from this that it is impossible to have one O motion only in the mechanism, and the other three U motions, or otherwise the sum of the four angles would be increased or diminished by 2π each revolution.

The law permits of there being four O motions. At two of the joints the angles will have to increase, and at the other two to diminish at the same rate, unless it be conceived possible for the rate of turning at one joint to be equal to that of the sum of the other three. The latter imagined movement will be found not to satisfy the conditions imposed by Law II.

Also Law I. permits the combination of three O's and one U if the rate of turning at one of the O's is on the average equal to the sum of the rates at the other two. Such a movement will be found to be not precluded by Law II. Further there may be two O's and two U's.

The capability of complete rotation at any joint will depend on the possibility of the adjacent links getting into the critical positions to lie along the same straight line.

1st. Away from one another.

2nd. Overlying one another.

Suppose Z^* to be capable of turning completely around relatively to T, then for Z

* Frequent reference to fig. 1 may be avoided if it is noticed that the small letters for the joints, and the capital letters which denote the intervening links, are in alphabetical order. Small letters $t v x z$ printed in italics are used to represent the length of the corresponding links.

and T to lie in the same straight line away from one another, the four links then forming a triangle, will require that

$$t + z = < v + x \dots \dots \dots (\alpha).$$

Also in order that they may be able to overlie one another, again forming a triangle with the two other links, the difference of t and z must not be less than the difference of v and x ,

$$t \smile z = > v \smile x \dots \dots \dots (\beta).$$

Therefore Law II. may be expressed as follows :—

In order that two consecutive links may be capable of turning relatively to one another, it is necessary that the sum of the lengths of those two links should not be greater, nor the difference less, than that of the other two links.

If these two criteria of complete turning motion be applied to each of two joints which are opposite one another, for example, the joint s between T and Z, and the joint w between V and X, it will follow that for complete turning at those joints,

$$\text{either } t = x \text{ and } z = v \dots \dots \dots (1),$$

$$\text{or } t = v \text{ and } z = x \dots \dots \dots (2).$$

The equalities (1) are also those which will permit of turning motions at the joints u and y , and if t does not exceed z , the equalities (2) will permit of turning motion at the joint u .

Thus, for four O motions, opposite links must be of equal length.

For three O motions and one U, the two links adjacent to the U must be equal to one another, and longer than the other two links, which must also be equal to one another.


It thus happens that in order that there may be O motions at two joints opposite one another, it will be necessary to have an O motion at at least one other joint, so that a combination consisting of two O motions alternating with two U's is not possible.

There being O motions at the two consecutive joints s and u , then, if t added to the length of one of the other links equals the sum of the other two, and, if added to each of the others the sum is less, or if t added to each one of the three gives a sum less than the sum of the two remaining, then, in either case, there can be U motions only at both w and y , but in the former case it will be possible for two complete revolutions to occur at s and u contemporaneously with one complete to and fro swing at each of w and y .

If the proportions are such as to satisfy neither set of conditions previously mentioned, then it will be possible for the motions at all the four joints to be U motions only.

For the motion at the joint of a quadrilateral, which, in the movement, causes a continually diminishing angle, it will be convenient to use a small *o*, to distinguish it from the motion which causes an increasing angle, which will be represented by a large *O*.

In the combination of four *O*'s, the two large *O*'s may alternate with the two small *o*'s, in which case the machine movement may be represented by a formula

written thus . For convenience in printing the letters may be put in line.

The links may be inserted thus —*O*—*o*—*O*—*o*—, and the end links imagined to be joined up, or the joining up may actually be exhibited thus *O*—*o*—*O*—*o*; or, in many

cases, it will be sufficient to leave the links themselves to the imagination, and show only the sequence of motions thus *OoOo*. This movement is that which is obtained by the mechanism known as “parallel cranks,” which is employed in locomotive engines to connect together two pairs of driving wheels by means of an outside coupling rod.

If the two angles which increase are in sequence, as would be expressed by *OOoo* or *OooO*, or *ooOO*, this movement is that belonging to so-called “anti-parallel cranks.”

If two complete rotations take place at one joint in the time occupied by one at another, or in the time taken by a complete swing to and fro, it will be convenient to express that fact by a doubled \odot , thus, or by O^2 .

Thus the combination of three *O*'s and one *U* may be written thus, oO^2oU , or when reversed thus, Oo^2OU . Considerations of symmetry indicate that the angular velocity at the two single *O*'s must be equal, and therefore, on account of the changing magnitude of the angle at the *U* joint, it will be alternately greater and less than half the angular velocity at the fourth joint.

As previously described, we may have the movement O^2o^2UU , and also $OoUU$.

Where a *U* motion occurs, as for example between *X* and *Z*, fig. 1, the construction of the links may be modified to the form shown in fig. 2, and yet precisely the same relative motion of all the links will be retained.

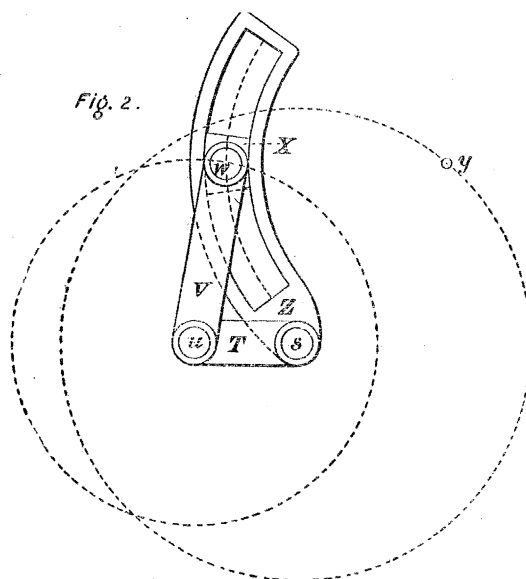
For the link *X*, of fig. 1, a block *X*, in fig. 2, is substituted, which swings about a centre *y*, which is the centre of curvature of the circular slotway formed in the link *Z*. *X* may be regarded as being guided by a portion of the surface of a large pin, instead of by the whole of the surface of a smaller one, the centre of the pins occupying the same relative position in each construction.

It is easy to conceive the centre of curvature *y* to be removed to a distance and the slotway to become less curved, until, in the limit when *y* is at an infinite distance, the slotway will be straight, and a sliding motion between *X* and *Z* will be substituted for a swinging motion. The resulting movement may be conceived to be that due to a four-bar mechanism, in which *X* and *Z* are infinitely long.

A sliding motion will be represented by the letter I.

This modification of the construction may be repeated where there is another U motion, but not where there is an O motion. The corresponding counterpart of an O motion would be a continuous slide in one direction, to provide for which a really indefinitely long link would be requisite.

The foregoing method of showing the derivation of an I motion from a partial turning motion is due to REULEAUX. It is here introduced for the purpose of showing how the two laws previously enunciated may be applied to mechanisms containing I motions.



In the application, a slide is to be regarded as a swing through a zero angle about an infinitely distant centre, and the adjacent links are to be imagined infinitely long, but having, possibly, a finite difference between them.

If in the combination oO^2oU an I is substituted for the U, it will become oO^2oI , in which the proportions between the length of the links previously ascertained must still be retained, that is, the two links joining the O's must be equal, and the other two links joined up by the I are to be conceived infinite and equal, which amounts to the construction being such that the line of slide is parallel to the line joining the axes of the two adjacent o motions. There being no angular velocity at the slide, the angular velocity of the O^2 motion must be exactly twice that of the two o motions.

This property, together with that of symmetry of motion, has caused this combination to be adopted in the design of a high-speed engine.

If a slide is substituted for a swing in the combination $OoUU$, it will become $OoUI$, many important examples of which are in every day use, as further described.

In the most important examples the line of slide is parallel to the line joining the axes of the adjacent U and O motions, which is equivalent to requiring the two links joined by the I to be imagined equal as well as infinite. But the line of slide may be oblique to the line joining the axes of U and O, short of an amount which would cause the distance between those axes measured perpendicularly to the line of slide to be equal to the difference in the lengths of the two finite links which join O and o and o and U. When the distance so measured is equal to the difference, a movement represented by O^2o^2UI may take place, and if the distance exceeds this difference, then all O motions are precluded. All this follows from Law II., and is consistent with the statement, p. 18, of the conditions requisite for $OoUU$ and O^2o^2UU .

Next, if, for the other swing, we substitute a second slide we obtain the movement $OoII$. In the most important examples of this the two lines of slide are at right angles to one another; but this is not necessary. It would be possible to increase the expressiveness of the formula by inclining the I's to the actual angle employed.

It is interesting to notice that the movement O^2o^2II is not possible, for with I's instead of U's the conditions requisite for O^2o^2UU could not be satisfied, since the sum of a finite link and an infinite link cannot be equal to the sum of two infinities. It may also be noticed that inasmuch as we cannot have $OUoU$ (p. 18), then, for the same reason, we cannot have $OUoI$, or $OIoI$.

From $UUUU$ we may derive by substitution $UUUI$, and $UUII$, and $UIUI$.

In the last mentioned all the links are to be conceived of infinite length, but the axes of the two U's are at a finite though variable distance apart.

The combination $UIII$ is not possible, for a slide is to be regarded as a swing through a zero angle about an infinitely distant centre, and there being no change in the angles of the imaginary quadrilateral at three of the joints, Law I. will preclude a change at the fourth joint.

The movement $IIII$ will be possible, but not only are all the links to be conceived of infinite length, as in $UIUI$, but there is no finite length whatever in the mechanism. This will permit of the movement being as indefinite as if there were five links of finite length joined by pins.

One of the links and slides may be suppressed, and the mechanism will then provide a definite relative movement represented by III , which may otherwise be shown with the I's inclined to one another, as the lines of slides are.

The fourteen combinations of the OUI motions which have been mentioned exhaust the list of the different possible ways in which they can be associated together in one simple plane mechanism.

The OUI motions permit of the possibility of the two pieces, which so move relatively to one another, being in contact with one another over an area which may be extended to any amount which may be deemed desirable.

From this property great durability against wear is provided, and this advantage, combined with that due to the facility with which the necessary surfaces of contact

can be formed, causes these motions to be so very largely employed in the composition of the machines found in ordinary use.

If the foregoing analysis be compared with that instituted by REULEAUX, to which it bears a close resemblance, it will be seen that REULEAUX conceives that the elementary essential components of *machines* are the *pairs of consecutive links* which are in mutual contact, whereas it is here proposed that the *relative motions* of consecutive links should be regarded as the essential elements or components of a *machine movement*.

Whilst the pairs of surfaces of contact of consecutive links should be formed to suit the kind of relative motion which those pieces are required to undergo, yet the forms of those surfaces do not themselves entirely govern the character of the motion.

REULEAUX assumes that the turning and sliding motions are entirely governed by the forms of the surfaces of mutual contact of the consecutive links, but shows that, to insure a more complex relative motion, a restraint is often imposed by means which are external to the two links which so move. Those additional means of constraint have to be included with that due to the forms of the surfaces of mutual contact, in the full conception of a complete pair. The apprehension of what exactly constitutes a pair is often extremely difficult.

REULEAUX, acting on this assumption, does not directly discriminate between a turning and a swinging pair, for there is no difference between the forms of the pairs of surfaces which are suitable for the O and U motions. Yet the difference in the two motions is most apparent, and very important, both kinematically and also from the practical engineer's point of view. So also, whilst the difference between the motions represented by a large O and a small o is easy to understand, and is of much practical importance, yet the necessary difference in the construction of the corresponding pairs is very intangible, and may even be non-existent.

Although REULEAUX supposes a mechanism to be made up of a number of pairs, yet, in some cases, he has to admit that the entire mechanism is needful to complete only one of the pairs. This he refers to as "chain closure." It will be seen that, if what REULEAUX calls a turning pair is differentiated into the pairs requisite for the three motions denoted by OoU, then, whenever either of those motions occurs, the whole mechanism will be needful for the completion of one pair contained in it. Thus, whilst saying that the whole is made up of a number of parts, yet each part complete will include the whole, and so the analysis fails.

In offering the proposed modification of the analysis of machines, the author desires to pay to REULEAUX a tribute of admiration for his work, and to gratefully acknowledge it as the source of his inspiration.

Although exception is taken to some of the premises of REULEAUX, yet the greater portion of his work will remain of undiminished value.

In designing or constructing a machine so that it may embody any one of the previously-mentioned combination of motions, the provision of suitable forms for the

surfaces of mutual contact and means of constraint against other motions will be found to exercise a judgment which is required to be extensively informed, the variety of possible forms and means being often so great. What REULEAUX describes under the headings "Complete and Incomplete Closure," "Force Closure," "Inversion of the Pair," "Chamber Pair," &c., will be found of great value in machine design; but, in the first study of the kinematics of machines in general, it will be found advantageous to avoid the consideration of these details, and to study the conditions necessary to permit of the contemplated motions, without being concerned with what is necessary to preclude other motions.

The great feature of REULEAUX'S theory is the enunciation, for the first time, of the principle known as "the inversion of a mechanism." By the aid of this, the family relationship between machines, which previously were thought to have little or nothing in common, is most clearly established; and all the knowledge which has been acquired relative to one becomes applicable to the consideration of the others.

The belief that the method of analysis here proposed is adapted to more perfectly and consistently express the effects of inversion, has been one of the chief encouragements to the author, in hoping that the adoption of his proposals will be advantageous.

It may be stated that the object aimed at in the design of a machine is to obtain a special kind of movement of one or more pieces, of which the machine is composed, relatively to one's self as a fixed observer or user of the machine.


Any one of the four links (if there are four) may be fixed relatively to the user—may be made, as it is called, the *frame link*, about which all the other pieces move; and we shall, in general, obtain a different kind of movement of those links with each change of the fixed link, the difference being such as to constitute practically a new machine movement, and yet all the four will be intimately related to each other, for they all have the same relative motions of the pieces composing them.

This idea of the change of the fixed or frame link is what is called "the inversion of a mechanism."

In order to assist in the description of the effect of inversion in causing a change in the machine movement, it will be convenient to adopt the term *primary piece*, originally proposed by RANKINE, for those pieces or links which are in sequence with the frame link and move in contact with it. (RANKINE must have been very near the discovery of the principle of inversion.)

If we invert any one of the previously mentioned mechanisms, and, by making some other link the frame link, find one or both of the new primary pieces to have a different motion to that which the previous primary pieces had, a new machine movement will have been derived; but if, as often happens, we find after inversion that both primary pieces move just like the previous primary pieces, no new movement will result, but only a repetition of a previous one. Thus there are not necessarily four different machine movements derivable by inversion from every four-linked mechanism.

The link which is selected for the frame may be represented as follows:—If the links and motions are shown in circuit the frame link may be shown by a

thickened line, thus,  or if printed in line, thus $\text{---O}^2\text{---o---I---o---}$, or thus.

$\text{O}^2\text{---o---I---o}$. Or, if the links themselves are left to the imagination, the link which unites the end letters of the line, may be understood to be the frame link,* so that O^2oIo means the same thing as either of the above three formulæ, and IoO^2o is an inversion which gives a new machine movement.

oIoO^2 and oO^2oI are inversions which give repetitions only of the previous movements.

An exhaustive list follows of all the possible different machine movements which can be derived by inversion from the combinations of the OUI motions previously enumerated.

They are divided first into four groups. These are sub-divided into fourteen combinations, and the fourteen combinations are still further divided into thirty-two so-called “inversions.”

When four different movements are derivable from a combination, each of the four may be regarded as an inversion of either of the others, and it will not be inappropriate in this case to say there are four inversions, or when only three or two different movements are obtainable, to say there are three or two inversions respectively. When only one movement can be obtained it will still be desirable to speak of it as one inversion.

* The author is indebted to one of the Referees, to whom this paper was submitted, for this suggestion and for some others.

GROUP O^4 , Containing 4 Turning Motions.

Combination.	Inversions.	Notes and examples.
$OoOo$	$oOoO$	Opposite links equal and parallel to one another. Gives one machine movement only, there being no change by "inversion." Mechanism, known as "parallel-crank," employed for coupling together two pairs of driving wheels of locomotive engines.
$OOoo$	$OOoo$	Opposite links equal but not parallel. Both cranks revolve in the same direction just as in the movement above.
	$OooO$	Two cranks revolve in opposite directions called anti-parallel cranks.

GROUP O^3 , Containing 3 Turning Motions.

Combination.	Inversions.	Notes and examples.
oO^2oU	oO^2oU O^2oUo	Mechanism consists of two equal short links in sequence and two equal longer ones. It is known as the Kite mechanism.
oO^2oI		There are two finite links of equal length and the line of slide is parallel to the line joining the axes of the two o's.
	oO^2oI	This movement has been used by BERNAY in the design of a steam-pump. It may be compared with the common crank-and-connecting-rod engine, in which the length of the connecting rod is equal to the crank. The length of stroke is four times that of the crank. This is supposed to be the advantage due to the use of this mechanism for this purpose.
	O^2oIo	This, used in duplicate, has been employed by PARSONS in the design of a high-speed engine. One primary piece, the crank shaft, revolves at twice the speed of the other primary piece which is the cylinder.

GROUP O^2 , Containing 2 Turning Motions.

Combination.	Inversions.	Notes and examples.
OoUU		The link between Oo must be such that when added to either one of the other three the sum must be less than the sum of the two remaining links.
	OoUU	In a beam-engine the portion consisting of the half-beam, connecting-rod, crank shaft, and frame of engine, affords an example.
	oUUO	Example: The drag-link coupling used for connecting two shafts which are parallel, but not in the same straight line.
	UOoU	In this case both primary pieces swing. The 4th inversion will give a repetition of the movement of the 1st.
OoUI		The line of slide must be at a less distance from the axis of the O motion than the difference in the length of the two finite links. In most practical examples of the use of this movement the deviation of the line of slide from the axis of O is zero, as in the following examples.
	OoUI	Example: The much used crank-and-connecting-rod engine.
	oUIO	Example: RIGG'S high-speed engine.
	UIOo	Example: The oscillating engine much used in paddle wheel steamers.
	IOoU	Example: STANNAH'S pendulum pump.
		Four different machine movements are derivable by inversion from this combination, and each has been adopted in the design of a steam engine.
O ² o ³ UU		The link between Oo added to one other link must equal the sum of the two remaining links; but when added to the others the sum must be less.
	O ² o ² UU o ² UUO ² UO ² o ² U	Two complete rotations occur at two joints, whilst one complete to-and-fro swing occurs at the other two. The 4th fixing gives a movement which is a repetition of the 1st.

GROUP O^2 , Containing 2 Turning Motions—(continued).

Combination.	Inversions.	Notes and examples.
O^2o^2UI	O^2o^2UI o^2UIO^2 UIO^2o^2 IO^2o^2U	In this combination the deviation of the line of slide is equal to the difference in the length of the two finite links. This is the limiting deviation possible compatible with O motions.
$OoII.$		Only one link is of finite length. The lines of the two slides may be inclined at any angle one to the other $>$ zero. In general, the angle is a right angle as in the following examples.
	$OoII$	Example: The Yoke pump, a compact engine, much used for feeding boilers. The relative motion between crank and piston is like that due to an infinitely long rod in a crank-and-connecting-rod engine. The motion is said to be "Harmonic."
	$oIIO$	This movement occurs in OLDHAM'S coupling employed to connect two shafts which are parallel but not in the same straight line. In this case the two shafts have a constant angular velocity ratio which is not the case with the drag-link coupling $oUUO$ previously quoted.
	$IOoI$	Every point in the link joining Oo moves in an ellipse. The mechanism has been used in an instrument for drawing ellipses. The 4th inversion would produce a movement which would be a repetition of the 1st.

GROUP U, Containing no Turning Motions.

Combination.	Inversions.	Notes and examples.
UUUU	UUUU	Whichever link is fixed both primary pieces will swing, there will therefore be no change in the movement by inversion. WATT'S rectilinear motion is an example of the use of this mechanism.
UUUI	UUUI UUUI	Only two movements by inversion, the other two inversions produce repetitions of the previous movements.
UUUII	UUUII UIIU IUUI	Three movements by inversion, the 4th being a repetition. There is no difference between the four links required for this combination and those required for OoII.
UIUI	UIUI	Whichever link is fixed, one primary piece will swing and the other slide, so that no new movement will be obtainable by inversion. Two of the infinite links adjacent to an I may be conceived to be equal, causing the line of that slide to be parallel to the line joining the axes of the two U's, as employed in RAPSON'S slide for steering large vessels, also in the compensating mechanism used in the WORTHINGTON steam pump, as pointed out, page 111, 'COTTERILL'S Applied Mechanics,' edition 1892. Both lines of slide may deviate from the line joining the axes of the U's. If the deviation of each is the same the mechanism has the remarkable property that in one position it may be locked. It appears to be quite new.
III	III	No change in the movement will occur on inversion. This is known as the wedge mechanism. It is employed in the cotter method of adjusting the brasses at the ends of a locomotive connecting rod when worn.

Although the OUI motions constitute the large majority of the relative motions of consecutive directly connected parts of machines, yet there are others, frequently found in use, in which the relative motion is of a more complicated character, for example the relative motion of the teeth of a pair of spur wheels. The relationship of machines in which there are such motions to the foregoing can be best explained by a reference to the series of machines shown in the adjoining five figures.

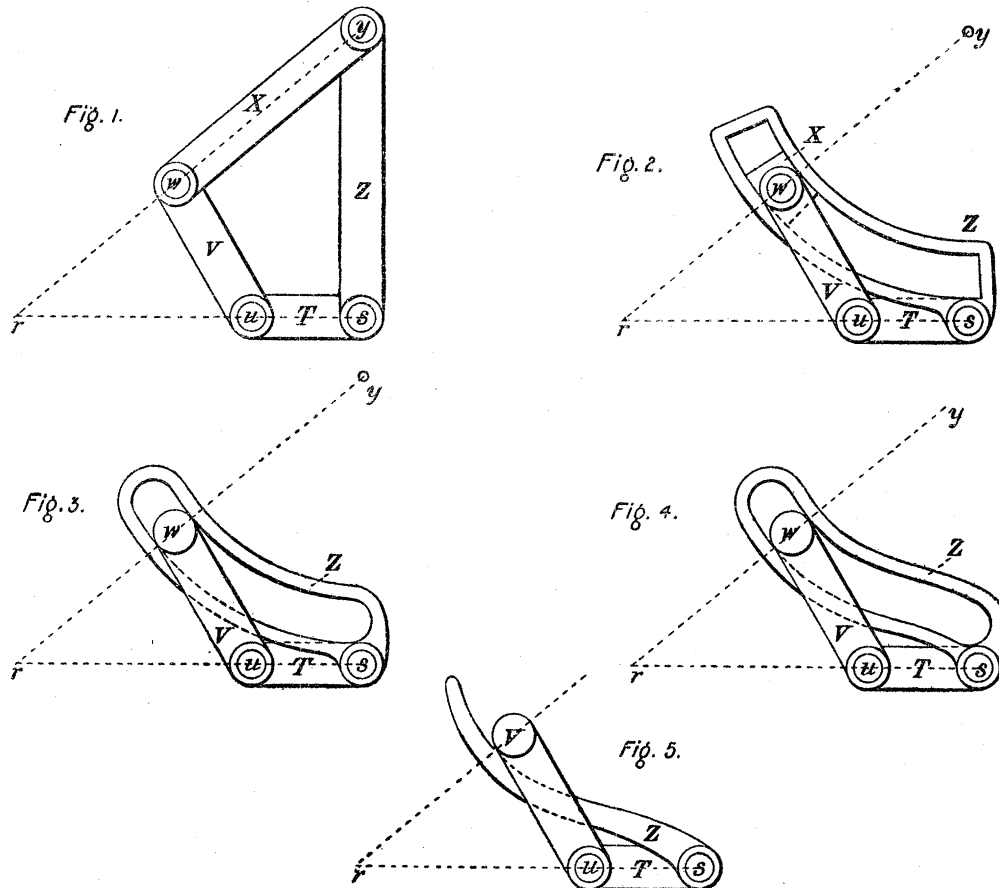
It has been previously shown that in the mechanism of fig. 2, p. 20, the same relative motions occur as in fig. 1, p. 16, though the construction is different. Suppose now the pin w to be enlarged sufficiently to fit the slotway in the link Z, as in fig. 3, p. 29. The block X may then be suppressed, and the same relative motion of the links T, V, and Z will be retained exactly as before.

The block X being omitted, it will no longer be necessary for the slotway to be curved to the arc of a circle. The radius of curvature may vary from point to point of its length; but, for definiteness of motion, it will be necessary for the width of the

slotway to remain constantly equal to the diameter of the pin, as shown in fig. 4. The centre of curvature of any portion of the slotway where the pin w is momentarily situated being y , the instantaneous relative motion of the parts will be the same as would be due to a four-linked mechanism of which the centres of the pins are at s , u , w , y . When the pin w moves to a place where the curvature is different, the relative motion will be the same as that due to a four-linked mechanism in which the lengths of X and Z are different from what they were previously.

If, in figs. 1, 2, 3, 4, the lines joining the centres y , w and s , u are produced to meet in the point r , it may be readily shown that

$$\frac{\text{The angular velocity of } V}{\text{The angular velocity of } Z} \text{ relatively to } T = \frac{sr}{ur}.$$



In the next consecutive position of the mechanisms 1, 2, and 3, the position of the point r will have changed, and with it the angular velocity ratio. In mechanism fig. 4, it will be possible, by a suitable curve for the slotway, for the intersection of the two lines to remain at a fixed point for a limited amount of motion, when, during

that period, the angular velocity ratio will remain constant, or the curvature of the slotway may be so designed as to cause a specially desired change in the angular velocity ratio.

An exceedingly important advantage is thus obtainable from the mechanism of fig. 4 as compared with 1 and 2 ; moreover, it is obtained by the use of one less link or moving piece. But these advantages are purchased at the cost of requiring that the consecutive links V and Z be in contact at a point, or along a line of points only instead of over an area, so that, if considerable pressure is transmitted from one link to the other, there is liability to undue abrasion and rapid wear.

If the action of the forces is such as to keep the link V always in contact with one side of the slotway, the other side may be omitted, as shown in fig. 5. In this case the pin w need not be cylindrical, but any shape.

The relative motion of V and Z in fig. 5 is comparable to that of two teeth of a pair of spur wheels or two cam surfaces. It will thus be seen to be an amalgamation of two adjacent OU or I motions.

The mechanism may be regarded as a mechanical artifice, whereby we get the movements which would be due to a four-linked mechanism in which the length of the links were capable, during the movement, of being varied in a desired way.

According to this point of view the same two geometrical laws will be applicable to mechanisms containing these motions. For example, from Law I. it will follow that the angular velocity of Z relatively to V must equal the algebraical difference of the angular velocities of Z and V relatively to T .

A convenient and expressive notation for such movements may be derived by suppressing the absent link and joining up the two simple adjacent OUI motions into one more complex. The letter W will naturally stand for an amalgamation of two U's, and the figure eight on its side, or the sign of infinity ∞ for two O's, or would be useful to indicate rolling motion.

The Greek letters Φ and Ψ are also available.

A mechanism consisting of two spur wheels, mounted in a link and in outside gear with one another, may be represented by OWO, and an inversion of it WOO. If the outside of one wheel gears with the inside of the other (an annular wheel), the formula should be OWo.

By the use of pulleys with ropes, belts, or other flexible links, another mechanical device is provided, whereby the angular velocity ratio in consecutive positions of the mechanism may be maintained constant, or caused to vary in a desired way. In this case the result may be considered to be achieved by a perpetual substitution of new links, exactly like the previous ones, and in the previous positions, when those have moved out of place, instead of regarding the movement to be continued by links of altered length, adapted to the new positions as we conceived to be the case with spur-wheels.

The relative motion of a belt to a pulley, which it wraps, is such that every point

describes an involute, so the letter C would, in a fairly graphical manner, represent this motion.

A mechanism consisting of two pulleys mounted in a frame joined by a crossed belt, may be represented by $OcCO$, or $oCco$. If an open belt is used the corresponding formula would be $OcCo$, a small c being used for wrapping, and a large C for unwrapping.

It is not easy, if possible, to make an exhaustive list of these mechanisms. What has been said will serve to indicate the method of the application of the previously mentioned principles in considering them.

It will be convenient to class them separately from the plane mechanisms made up of the OUI motions.

Spherical Mechanisms.

The next division of machines to be described will be those in which the simple motions OU only are employed, but in which the axes, instead of being parallel to one another, are so inclined that they all meet in one point.

Referring to the original mechanism, fig. 1, p. 16, suppose the links to be bent so that they lie on or parallel to the surface of a sphere, the axes of the pins will all be radii of the sphere, and if produced will meet at the centre, and the centre lines of the links will be great circles of the sphere.

REULEAUX has shown that the previously described mechanisms, or some of them, have their counterpart in spherical movements; but he omitted to notice an important exception which forms the foundation for a distinct division of machines, in which helical motions are employed.

There are other points of interest and importance, not observed by REULEAUX, which follow from the application of the geometrical laws which govern the association of the OU motions in a spherical mechanism. For such movements Law I. must be modified as follows:—

The sum of the four angles of the spherical quadrilateral varies, having a value of 3π for a maximum, and 2π for a minimum.

Law II. will remain as before.

It will follow that each one of the combinations of the OU motions in plane mechanisms will have its counterpart in spherical mechanisms.

REULEAUX has further pointed out that if a mechanism containing a sliding motion is adapted to a sphere, then, instead of a sliding motion along a curved link, exactly the same relative motion could be produced by the swinging of one link about one of the poles of the sphere, of which the curved line of slide is the equator. One of the two links joined by the slide would then be required to have a length equal to that of a quadrant of a great circle. The other may be equal to it or not equal, being greater or less at pleasure, for either pole may be selected for the axis of swing.

Also all the combinations of the I with OU motions previously detailed have their counterpart in spherical mechanisms.

REULEAUX has compared one of the best known spherical mechanisms, known as HOOKE'S joint, with its plane counterpart, OLDHAM'S Coupling, p. 27. In the latter there are three links which are supposed to be equal and infinite. In the former there are three links, the lengths of which are each equal to a quadrant, the fourth being shorter. In HOOKE'S joint it is well known that the angular velocity ratio of the connected shafts is not constant, whereas in OLDHAM'S Coupling it is.

It is interesting to notice that these two facts immediately follow from Law I., the sum of the four angles of the spherical quadrilateral not being constant.

In spherical mechanisms the relative lengths of the links may be represented by the magnitude of the angles at the centre of the sphere which they subtend. This suggests that a spherical movement may be represented and distinguished from a plane movement by inserting the value of the angles between the letters O and U, which represent the motions.* Thus for examples :—

The spherical counterpart of OoOo is $\beta O\alpha o\beta O\alpha o\beta$, expressing the fact that opposite links are equal to one another.

Of oO²oU, the spherical counterpart is $\beta O\alpha O^2\alpha o\beta U\beta$ where $\beta > \alpha$.

Of oO²oI it is $\frac{1}{2}\pi O\alpha O^2\alpha o\frac{1}{2}\pi U\frac{1}{2}\pi$. $\alpha < \frac{1}{2}\pi$; and so on for all the other plane mechanisms previously enumerated, with one exception.

The movement III has no spherical counterpart capable of movement, as it will be a spherical triangle, though there is a spherical counterpart to the movement IIII.

It is interesting to notice that the reason which precluded the existence of the combination UIII in plane mechanisms, does not hold in the case of its spherical counterpart, for the sum of the four angles of the spherical quadrilateral is capable of variation. With these exceptions the combinations not possible in plane mechanisms are also impossible in their spherical counterparts.

An enumeration will show that there are only six different ways of combining the OU motions in a spherical mechanism, and out of these we can get only twelve different movements by inversion.

Besides these there are the spherical counterparts of those plane mechanisms which contain in their composition other motions than OUI. The most notable of them is the mechanism consisting of a pair of bevil wheels mounted in a frame, the formula for this would be $O\alpha W\beta O(\dot{\alpha} + \dot{\beta})$.

* The formulæ for plane mechanisms may be made to give information about the length of the links by inserting figures between the letters which represent the motions. Thus O2'o8'UI may be considered to represent a crank and connecting-rod engine of 4' stroke, the length of the connecting-rod being twice the stroke, and the line of stroke passing through the axis of the rotating crank. If the line of stroke were to deviate by the amount of, say 1', the fact could be indicated in the formula by O2'o8'U1'I, one of the infinite links being 1' greater than the other.

Cylindrical Mechanisms.

Whilst the plane mechanism III cannot be adapted to the surface of a sphere, it can be fitted to the surface of a cylinder, and it is the only one of the OUI mechanisms which can.

If such a triangle of slides be imagined to be wrapped around the surface of a cylinder, one at least of the slides will have to become helical. Two or even three may be.

In this way we are led to the very important Third Division of machines containing screw motions.

The projected view of a helix has a shape, much like the letter V, so this letter will suggestively stand for helical motion.

In this division there are the following combinations :—

- UVI with three inversions. The common screw press is one example.
- VVI with two inversions. The differential screw press and micrometer are examples.
- VVU with two inversions, and
- VVV one inversion only.

In total, four combinations, containing eight inversions.

But not only is the plane mechanism III capable of being adapted to a cylinder, but also IWI. If for one of the slides of the former mechanism a curved slotway of uniform width fitted by a cylindrical pin were substituted, a mechanism represented by the latter formula would be obtained in which the relative velocities of sliding would not be constant, but vary from point to point. If, further, the resulting mechanism be conceived to be wrapped around a cylinder, we should have one helix of varying pitch. In order that the pin may fit such a helical groove, it will be necessary to restrict the contact between the two to that of one or a pair of points only. Such a mechanism is known as a cylindrical cam, and there are many applications of it in practice.

Adopting the letter H for the motion with the varying-pitch helix, the following four combinations are possible, and they have eleven different inversions, viz. :—

- UHI with three inversions.
- VHI " " "
- VHU " " "
- VHV " two "

No other combinations belonging to this division or sub-division appear to be possible.

Skew Mechanisms.

It is proposed to group all the remaining simple mechanisms in a fourth division. They may be described generally as consisting of those in which the axes of the elementary motions neither meet nor are parallel.

These may be referred to under the name of skew mechanisms.

The general case may be described as consisting of a link T secured to a shaft, which turns or swings in bearings provided in a piece V, which also provides bearings for another shaft which is neither parallel to nor meets the former shaft T. This second shaft has a link X secured to it so that it also turns or swings relatively to V. The two links T and X, which are either cranks or levers, are joined together by a link Z, which has ball and socket connections with T and X.

With such a construction the movement will be of a definite character, and the mechanism will be capable of serving the purpose of a machine. By imagining either or both links T and X to be infinitely long, a slide or slides may be substituted for one or both of the turns or swings.

Thus a variety of combinations are possible, and each one will be capable of a certain number of inversions.

In the formulæ for these it is proposed to use the Greek letter Θ for the movement at the ball and socket connection.

A certain limited number of OUI Θ combinations will be possible of which an exhaustive list could be prepared.

Besides these another set of skew mechanisms will result from suppressing one of the links and amalgamating the two adjacent simple motions. The best known examples of such a combination is seen in the use of a pair of skew-bevil-wheels and in the worm-and-worm-wheel. The relative motion between two such pieces may be represented by the Greek letter Ω .

Summary of Simple Mechanisms.

The method of classification, according to the proposed scheme, may be summarized as follows :—

All simple machine movements may be ranged in four divisions, viz. :

1. Consisting of plane mechanisms in which the pieces move in or parallel to one plane.
2. Spherical mechanisms in which the pieces move in or parallel to the surface of a sphere.
3. Cylindrical mechanisms in which the pieces move in or parallel to the surface of a cylinder ; and
4. Skew mechanisms, being those in which the axes of the turning or swinging motions neither meet nor are parallel.

The mechanisms in each of these divisions may be classed in two sub-divisions :

S being the sub-division in which each pair of consecutive links is in contact one with the other over a surface.

P being those in which one or more pairs of consecutive links are in contact with one another at a point or along a line of points only.

The mechanisms of Sub-division S of Divisions 1 and 2 will consist of those in which the OUI motions only are used.

Those of Division 3 will include the V or helical motion ; and

Those of Division 4 will include the motion Θ , requiring the use of a ball-and-socket joint.

To the pairs of links which have the relative motions denoted by OUIV, REULEAUX has given the name "Lower Pairs."

REULEAUX claimed two characteristics for lower pairs, viz. :

1. Definiteness of motion derived from the forms of the surfaces of mutual contact and depending on nought else.

2. The possibility of distributing the contact of consecutive links over an area which may be extended as much as desired, the contact not being confined to a point or a line of points as in "Higher Pairs."

It has been shown that if the motions represented by Oo and U are required to be differentiated from one another, REULEAUX's so-called turning pair cannot possess the first characteristic. (There is scarcely a mechanism in which the nature of the motion between two consecutive links does not depend on the other links.)

The second characteristic is of considerable value in relation to the liability to abrasion and wear ; but the advantage of greater immunity against wear has to be purchased at the cost of a more complicated construction, and a more restricted character of movement.

As the first characteristic cannot be secured, the author proposes to adopt the second only for the criterion as to whether a mechanism should be regarded as belonging to Sub-division S or P.

Therefore, in Division 4, the motion Θ requiring the ball-and-socket joint should be included in the motions of Sub-division S. REULEAUX considered the ball-and-socket joint a "higher pair."

If the motion of any two consecutive pieces of a mechanism differs from the motions OUIV and Θ , the mechanism will belong to Sub-division P.

Next, the mechanisms included in either of the eight sub-divisions may be separated into sections as numerous as the combinations of the various elementary motions, which will satisfy the governing geometrical laws.

For example, in Sub-division 1s, it has been shown that there are fourteen, and only fourteen, possible combinations. (These have been placed in four groups, chiefly to aid the memory in enumerating them.) In the Sub-divisions 2s there are only six distinct combinations, and in 3s there are four. It is probable that in the other sub-divisions the number of possible combinations will be finite.

The next step in the discrimination of one machine movement from another will consist in distinguishing the various different movements derived from any one combination by inversion.

Compound Mechanisms.

In most of the important machines it will be found that there are more than four, and often many more than four, separately moving pieces and yet the motions are of a perfectly determinate character. It is left to explain how this is so, and how the previous considerations apply to such machines.

In practically every one of these machines it will be found, on examination, that there is a mechanism consisting of not more than four pieces associated together in one continuous chain. The pieces or links of this chain, which for reference may be called A, will therefore have a perfectly definite motion, either relatively to the user of the machine, if one of the links is the frame, or only relatively to one another if neither of those links is the frame.

The compound machine will be formed by one or more of the links of A being combined with other pieces to form a second mechanism B.

Suppose one link only of A to be joined up in chain with three fresh pieces to form the second continuous or closed chain B, then the movement of B will be independent of A, for we may imagine the link which is common to the two to be the frame link, in which case A may be in movement whilst B is at rest, or *vice versa*, or both may be in motion at the same time.

But more often it will be found that two of the links of A are adopted to compose with two new links the second mechanism B. In this case, a movement of A cannot take place without causing a definite movement in the two new links which, together with two of A, form the second mechanism B. Of the six links, which in this way have definite relative motions, any two may be adopted to form, in conjunction with two additional links, a third mechanism C producing a machine consisting of eight pieces, having a perfectly definite motion relatively to one another, and so on.

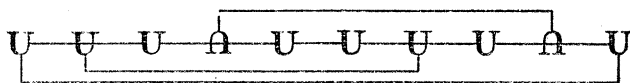
Compound mechanisms may be conceived and constructed in which there is a definite relative motion, but in which there is not a four-linked continuous chain of pieces.

The most simple construction of this character may be described as consisting of a continuous chain of six links, of which two pairs of links, which are opposite to one another, are linked together by the addition of two more links. In this case the shortest continuous chain will contain five links. Of the eight links which make up the complete mechanism, four will form a part of three five-linked chains, and four a part of two.

The simultaneous control due to the two or three partial restraints are sufficient to determine a definite motion.

In this mechanism there will be U motions at all the ten joints, and there will be four bars, each containing three joints. Each of these bars will be continuous

through a joint, and in the formula may be represented, coupled to three other links, thus —U—U—U— , in which case the formula for the whole mechanism may be shown so



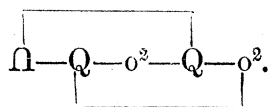
The author has been unable to discover any actual machine in which determinate motion has been derived from such a method of accumulation of partial constraints.

Passing to the compound mechanisms previously referred to, four degrees of complexity in the conjunction of two interdependent simple mechanisms may be specified.

Conjunction I, when two adjacent links, T and V for example, of a mechanism A are adopted, in their exact length, to form with two new links, X_2 and Z_2 , the second mechanism B. In this case the new link X_2 is attached to V at the same joint, w , at which X_1 is coupled, and Z_2 as well as Z_1 are united to T at the joint s . In this we have a third mechanism, B', consisting of the four links, $X_1Z_1Z_2X_2$, united at the joints y_1sy_2w . Supposing U motions at all the joints, the formula for a compound mechanism

so conjoined may be written thus, $\Omega\text{—U—U—}\Omega\text{—U—}$. An example of this will be

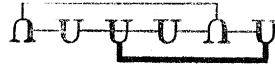
found in what is called a double Kite mechanism (see p. 25) consisting of four equal links forming a parallelogram, and two links longer but equal to one another, joined together at one end, and at the other ends united to two opposite corners of the parallelogram. Such a construction forms a part of the mechanism of PEAUCELLIER'S straight-line motion. If the additional links of PEAUCELLIER'S mechanism are omitted, it will be possible, with each Kite, to have the movement UOo^2O as before explained (p. 19). Now, if three links unite at a joint and each of two of them have an O motion relatively to the other then, relatively to one another, the motion must be either U or O^2 . A new symbol is therefore wanted to stand for a motion which may be either O, U, or O^2 , which it is, to be determined by the context. The letter Q suggests itself, and accordingly the formula for the double Kite would be



All the previously-mentioned conditions as to the associations of the OUI motions in one chain, and the proportions of links requisite for any combination, will hold good of mechanism B, as well as A. Thus the letter Q in the mechanism above stands for O in each Kite mechanism, and for O^2 in the parallelogram mechanism. It will frequently happen that whilst the proportions of A will permit of O motions

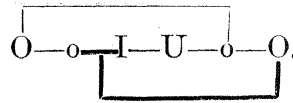
at two or more joints, those of B will preclude it, in which case the motions will be restricted to U motions.

In Conjunction II., two adjacent links of A are adopted, but only one in its full length. Thus the two new links which are added to form B are connected to the two chosen ones of A at, say, the joints s and w_2 . In this case the third mechanism B' will consist of five links coupled at the joints w_2, w_1, y_1, s, y_2 . Supposing, as before, U motions only, the formula for such a conjunction may be written thus,



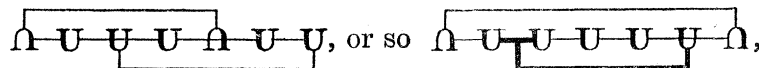
The link shown by the thick line is as explained (p. 24), intended to represent the frame link. The preceding formula will represent a portion of the mechanism of a beam engine consisting of the frame of the engine, the half beam, and the parallelogram and radius rod of WATT'S parallel motion.

Another example of Conjunction II. may be quoted for the purpose of showing how the formula may be written when there is an I motion in the mechanism. The portion of a locomotive engine consisting of frame, piston, connecting-rod, crank, outside coupling-rod, and the crank of a second driving wheel, will have a movement which may be written thus,



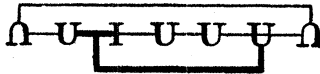
The letter I does not lend itself to showing whether the two adjacent links form one rigid piece or not; but if it be remembered that a slide is to be regarded as a swing about an infinitely distant joint, it will be seen that two lines meeting at an I cannot represent one link continuous through the joint, but must always be understood to indicate two separate links. Thus, no difficulty will occur in the use of the letter I.

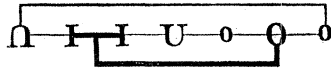
In the next, Conjunction III., neither of the two adjacent links of A, which are adopted, are used with their original length in mechanism B, so that each of these two links possess three joints. Thus, the mechanism B' will consist of six links connected at the joints $w_2, w_1, y_1, s_1, s_2, y_2$. Supposing U motions only, the formula for this conjunction may be written thus,



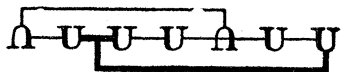
when one of the three-joint links is the frame link.

Of this, two examples may be quoted with which the reader will be sufficiently familiar to dispense with illustrations. One is the mechanism of RICHARD'S indicator for taking diagrams of steam pressure in engines, the formula for which may be

written thus  and the other is the common crank and connecting-rod engine with eccentric and slide-valve, the formula for which is

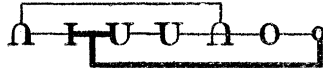


In Conjunction IV. it is the two opposite or alternate links of A to which the two additional links are coupled in the formation of mechanism B. The two selected alternate links of A must each be provided with three joints. Thus, mechanism B, as well as B', will consist of five links. The formula may be written thus

 when one of the two three-jointed links is the frame link.

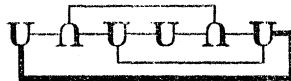
The mechanism of the CROSBY steam indicator affords an example of this conjunction. The same formula will represent the movement if an I be substituted for the third U from the left.

The ATKINSON gas engine may be quoted as another example, the formula being

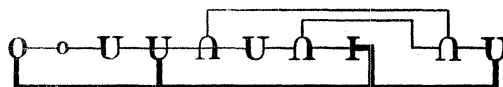


If three or more simple mechanisms are compounded, the varieties in the ways in which they may be conjoined are much more numerous. It is not desirable to set them forth here. Enough has been said to indicate the lines which it is proposed should be followed in systematizing them. The formulæ of some well-known machines, consisting of three or more simple mechanisms, are added.

PEAUCELLIER'S straight-line mechanism is an example of a three-fold conjunction, and may be written thus,

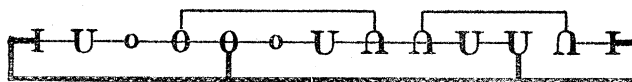


The Beam engine, with WATT'S parallel motion, and engine piston driving a rotating shaft is an example of a four-fold conjunction, it may be written thus,



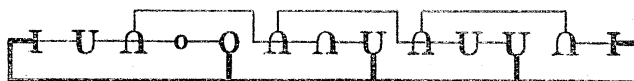
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The formula for a reversing engine, with STEPHENSON'S link motion, is



The object of the mechanism is to get a correct relative movement of the two sliding pieces, piston and valve, which are represented by the links adjacent to the I's at the two ends of the formula.

With this may be compared the formula



for an engine with Joy's valve gear, which is very differently constructed for the same purpose.